Global Attractor and Exponential Attractor for Higher-Order Kirchhoff Equations with Fading Memory

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Abstract: In this paper, we study the initial and boundary value problems of higher order Kirchhoff equation with higer-order memory term. Firstly, under approprate assumptions, we use Galerkin finite element method and a priori estimation to prove the existence and uniqueness of the global solution of this kind of equation in detail; and use contraction function menthod to prove the asymptotics of the solution semigroup, and then we get the existence of the global attractor ; in the end, we discuss the exponential attractor of a class of equations, and the finite fractal dimension of the global attractor is obtained.

1. Introduction

In this paper, we study the initial boundary value problems for the following higher order Kirchhoff equations with decaying memory terms

$$\begin{cases} u_{tt}+(-\Delta)^m u_t+\phi\left(\left\|\nabla^m u\right\|^2\right)(-\Delta)^m u+\int_0^\infty g(s)(-\Delta)^m u_t(t-s)ds+f(u)=h(x), \\ u=0,\ \dfrac{\partial^i u}{\partial v^i}=0,\ i=1,2,\cdots,m-1,\ x\in\Gamma,\ t\ge 0, \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x),\ x\in\Omega. \end{cases} \tag{1}$$

Where Ω is a bounded domain in R^3 with smooth boundary Γ, v is the outer normal vector on boundary Γ, $m\ge 1, u_t(x,-t), t\ge 0$ is a prescribed past history of u_t, $g(s)$ is the memory core, $h(x)$ is an external force, $\phi(\bullet)$ is a nonnegative real valued function, $f(u)$ is a nonlinear source term.

In 1883, Kirchhoff established the Kirchhoff equation for describing the cross-section motion of elastic rod,Since then, there have been many in-depth studies on Kirchhoff type equations, and various rich results have been obtained. Yang Zhijian and Cheng Jianling[1]proved the long-time behavior of the solution of the following Kirchhoff type equation $u_{tt}-M\left(\left\|\nabla u\right\|^2\right)\Delta u-\Delta u_t+g(x,u)+h(u_t)=f(x)$ with strong damping term,With two different methords,it proves that the related continuous semigroup $S(t)$ posseses in phase space $X=\left(H^2\cap H_0^1\right)\times H_0^1$ a global attractor.,At the end of the paper,an example is shown,which indicates the existence of nonlinear functions. Guoguang Lin, Penghui Lv and Ruijin Lou[2] studied the dynamic behavior of a class of generalized nonlinear Kirchhoff Boussinesq type equations, and proved the existence of exponential attractors and inertial manifolds. HuaChen and Gongwei Liu[3] studied the initial boundary value problem of nonlinear Kirchhoff type wave equation with damping and memory terms. Under certain conditions, the existence of local and global existence and exponential decay were obtained. When the weak damping term was nonlinear, the energy increased exponentially with time; when the weak damping term was linear, the energy blew up. there have been a lot of impressive literatures [4-6].

With the deepening of research, scholars began to study the related properties of more generalized Kirchhoff type equations, such as the related properties of higher order Kirchhoff type equations Ye Yaojun and Tao Xiangxing [7] studied the initial boundary value problem for a class of higher order Kirchhoff type equations with nonlinear dissipative term. By constructing stable sets, they discussed the existence of global solutions to the problem, and used Nakao's difference inequality to establish the decay estimate of solution energy. And they proved that under the condition of positive initial energy, the solution will blow up in finite time, and the life interval of the solution is estimated. Lin Guoguang and Li zhuoxi [8] studied the initial boundary value problem for a class of higher order Kirchhoff type equations with nonlinear nonlocal source term and strong damping term. Firstly, the existence and uniqueness of the solution were proved by galerlin finite element method,Furthermore, a family of global attractors is obtained. The Hausdorff dimension and fractal dimension of the global attractor family are finite. More literatures on higher order Kirchhoff type equations can be found in[9-11]. At present, there are few studies on the higher order Kirchhoff equation. In this paper, we will study the more characteristic Kirchhoff equation with decay memory term, and discuss the global attractor and exponential attractor of this kind of equation.

In order to study smoothly, we first define all kinds of spaces and symbols,

Without losing its generality, The inner product and norm of definition $L^2(\Omega)$ are respectively $(\bullet,\bullet)$ and $\|\bullet\|$,In particular

$$H=L^2(\Omega), V_m=H_0^m(\Omega)=H^m(\Omega)\cap H_0^1(\Omega),$$

The corresponding inner product and norm are:

$$(u,v)_{V_m}=(\nabla^m u,\nabla^m v),\ \|u\|_{V_m}=\|\nabla^m u\|_{L^2}.$$

Next, we give the history space:

$$\mathrm{E}=L_\mu^2(R^+;V_m)=\left\{\eta:R^+\to V_m\left|\int_0^\infty \mu(s)\|\eta(s)\|_{V_m}^2\,ds<\infty\right.\right\},$$

Obviously, the space is a Hilbert space with inner-product and norm

$$(\eta,\zeta)_{\mathrm{E}}=\int_0^\infty \mu(s)\left(\int_\Omega \nabla^m\eta(x,s)\bullet\nabla^m\zeta(x,s)\,dx\right)ds,\ \|\eta\|_{\mathrm{E}}^2=\int_0^\infty \mu(s)\|\eta(s)\|_{V_m}^2\,ds.$$

At the same time, there is a general Poincare inequality: $\lambda_1\|\nabla^r u\|^2\le\|\nabla^{r+1}u\|^2$,where λ_1 is the first eigenvalue of $-\Delta$. For brevity, we use the same letter C denote different positive constants, and $C(\bullet)$ denote positive constants depending on the quantities appearing in the parenthesis.

Assume that

(I) The nonlinear function $f\in C^1(R)$ satisfies the following conditions

$(F_1)\ \lim\limits_{|s|\to\infty}\dfrac{F(s)}{s^2}\ge 0$;

$(F_2)\ \liminf\limits_{|s|\to\infty}\dfrac{sf(s)-\rho F(s)}{s^2}\ge 0$,where $0<\rho<2$;

$(F_3)\ |f(s)|\le c_1(1+|s|)$,where $c_1>0$;

and $F(s)=\int_0^s f(\tau)\,d\tau$.

(II) $\phi\in C^1(R^+),\ \phi'\ge 0,\ \phi(0)=\phi_0>0$.

(III) Memory kernel function $g(\bullet)\in C^2(R^+),\ g'(s)\le 0\le g(s),\ g(\infty)=0,\ \forall s\in R^+$,and $\mu(s)=-g'(s)$ satisfies

$(G_1)\,\mu \in C^1(R^+) \cap L^1(R^+),\ \mu'(s) \le 0 \le \mu(s),\ \forall s \in R^+,$

$(G_2)\,\mu_0 = \int_0^\infty \mu(s)\,ds > 0,\ \mu'(s) + \mu_1\mu(s) \le 0,\ \forall s \in R^+,\ \mu_1$ is a positive constants.

Equation (1) is transformed into a definite autonomous dynamical system. Here we follow the presentation by[12, 13],accordingly,one defines a new variable η that corresponds to the relative displacement history. That is,

$\eta = \eta^t(x,s) = \int_0^s u_t(x,t-\tau)\,d\tau,\ (x,s) \in \Omega \times R^+, t \ge 0$. (2)

By formal differentiation we have

$\eta_t^t(x,s) = -\eta_s^t(x,s) + u_t(x,t),\ (x,s) \in \Omega \times R^+, t \ge 0$. (3)

Therefore problem becomes:

$$\begin{cases} u_{tt} + (-\Delta)^m u_t + \phi\left(\|\nabla^m u\|^2\right)(-\Delta)^m u + \int_0^\infty \mu(s)(-\Delta)^m \eta^t(s)\,ds + f(u) = h(x), \\ \eta_t = -\eta_s + u_t,\ (x,t,s) \in \Omega \times R^+ \times R^+,\quad (x,t) \in \Omega \times [0,\infty). \end{cases} \quad (4)$$

With boundary condition:

$u = \dfrac{\partial^i u}{\partial v^i} = 0,\ (x,t) \in \Gamma \times R^+,\ \eta = \dfrac{\partial^i \eta}{\partial v^i} = 0,\ x \in \Gamma,\ t \ge 0.$ (5)

And initial conditions:

$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \eta^0(x,s) = \eta_0(x,s),\ \eta^t(x,0) = 0.$ (6)

2. Existence and Uniqueness of Solutions

Let $X = V_m \times H \times \mathrm{E}$, $z_0 = (u_0, u_1, \eta_0)$, $z = z(t) = (u(t), u_t(t), \eta^t(s))$.

Lemma2.1 Let$(\mathrm{I})-(\mathrm{III})$be established,and$h \in H,\ z_0 \in X$，$\varepsilon$ is an appropriate small normal number,then z determined by problems(4)-(6)satisfies the following properties:

$\|u_t + \varepsilon u\|^2 + \|\nabla^m u\|^2 + \|\eta^t\|_{\mathrm{E}}^2 \le R_1^2, (t \ge T_0)$, (7)

$\int_0^t \|\nabla^m u_t(\tau)\|^2 d\tau + \dfrac{\mu_1}{2}\int_0^t \|\eta^\tau\|_{\mathrm{E}}^2 d\tau \le R_2$. (8)

Proof:Let$v = u_t + \varepsilon u$,Taking$H$-inner product by$v$ in(4),we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[\|v\|^2 + \varepsilon^2\|u\|^2 - \varepsilon\|\nabla^m u\|^2 + \int_0^{\|\nabla^m u\|^2}\phi(s)\,ds + 2\int_\Omega F(u)\,dx\right] + \|\nabla^m v\|^2 - \varepsilon\|v\|^2 + \varepsilon^3\|u\|^2 \\ &-\varepsilon^2\|\nabla^m u\|^2 + \varepsilon\phi\left(\|\nabla^m u\|^2\right)\|\nabla^m u\|^2 + \varepsilon(f(u),u) + \left(\int_0^\infty \mu(s)(-\Delta)^m \eta^t(x,s)\,ds, v\right) = (h,v), \end{aligned} \quad (9)$$

By $(\mathrm{I})-(\mathrm{III})$,Poincare inequality and Holder Inequality,we have

$\int_\Omega u f(u)\,dx \ge \rho\int_\Omega F(u)\,dx - \rho_1\|u\|^2 - C(\rho_1)|\Omega|$,其中$\rho_1 > 0$, (10)

$\varepsilon\phi\left(\|\nabla^m u\|^2\right)\|\nabla^m u\|^2 \ge \varepsilon\phi_0\|\nabla^m u\|^2 \ge \varepsilon\int_0^{\|\nabla^m u\|^2}\phi(s)\,ds$, (11)

$$\begin{aligned} &\left(\int_0^\infty \mu(s)(-\Delta)^m \eta^t(x,s)\,ds, v\right) = \left(\int_0^\infty \mu(s)(-\Delta)^m \eta^t(x,s)\,ds, u_t + \varepsilon u\right) \\ &\ge \frac{1}{2}\frac{d}{dt}\|\eta^t\|_{\mathrm{E}}^2 + \frac{\mu_1}{4}\|\eta^t\|_{\mathrm{E}}^2 - \varepsilon^2\frac{\mu_0}{\mu_1}\|\nabla^m u\|^2, \end{aligned} \quad (12)$$

$(h,v) \le \|h\|\|v\| \le \dfrac{1}{4}\|\nabla^m v\|^2 + \lambda_1^{-m}\|h\|^2$, (13)

Substituting (10)-(13)into(9),we get

$$\frac{d}{dt}H_1(t)+K_1(t)\le 2\varepsilon C(\rho_1)|\Omega|+2\lambda_1^{-m}\|h\|^2, \text{ (14)}$$

where

$$H_1(t)=\|v\|^2+\varepsilon^2\|u\|^2-\varepsilon\|\nabla^m u\|^2+\int_0^{\|\nabla^m u\|^2}\phi(s)\,ds+\|\eta^t\|_{\mathrm{E}}^2+2\int_\Omega F(u)\,dx,$$

and

$$K_1(t)=2\left(\frac{3}{4}\|\nabla^m v\|^2-\varepsilon\|v\|^2+(\varepsilon^3-\varepsilon\rho_1)\|u\|^2+\varepsilon\int_0^{\|\nabla^m u\|^2}\phi(s)\,ds-\varepsilon^2\left(1+\frac{\mu_0}{\mu_1}\right)\|\nabla^m u\|^2\right)$$

$$+\frac{\mu_1}{2}\|\eta^t\|_{\mathrm{E}}^2+2\varepsilon\rho\int_\Omega F(u)\,dx,$$

Select the appropriate small $\varepsilon>0$,then

$$H_1(t)\ge\kappa_1\left(\|v\|^2+\|\nabla^m u\|^2+\|\eta^t\|_{\mathrm{E}}^2\right)-C(\rho_1)|\Omega|,$$

and

There is a sufficient small normal number $\alpha_1>0$,such that

$$K_1(t)\ge\alpha_1 H_1(t),$$

(14)become

$$\frac{d}{dt}H_1(t)+\alpha_1 H_1(t)\le 2\varepsilon C(\rho_1)|\Omega|+2\lambda_1^{-m}\|h\|^2, \text{ (15)}$$

According to Gronwall inequality, we get

$$H_1(t)\le H_1(0)e^{-\alpha_1 t}+\frac{2\varepsilon C(\rho_1)|\Omega|+2\lambda_1^{-m}\|h\|^2}{\alpha_1}, \text{ (16)}$$

Therefore, there are normal numbers R_1 and $T_0\ge 0$,such that

$$\|v\|^2+\|\nabla^m u\|^2+\|\eta^t\|_{\mathrm{E}}^2\le R_1^2,\ (t\ge T_0). \text{ (17)}$$

Taking H inner product by u_t in(4),we obtain

$$\frac{1}{2}\frac{d}{dt}\left[\|u_t\|^2+\int_0^{\|\nabla^m u\|^2}\phi(s)\,ds+\|\eta^t\|_{\mathrm{E}}^2+2\int_\Omega F(u)\,dx-2(h,u)\right]+\|\nabla^m u_t\|^2+\frac{\mu_1}{2}\|\eta^t\|_{\mathrm{E}}^2\le 0, \text{ (18)}$$

Integrating(18) over $(0,t)$,and by(16),there is a normal number R_2,such that

$$\int_0^t\|\nabla^m u_t(\tau)\|^2\,d\tau+\frac{\mu_1}{2}\int_0^t\|\eta^\tau\|_{\mathrm{E}}^2\,d\tau\le R_2, \text{ (19)}$$

Lemma 2.1 is proved.

Theorem2.2(Existence and uniqueness of Solutions) Let $(\mathrm{I})-(\mathrm{III})$ be established,and $h\in H$, $z_0\in X$,Then problem (4)-(6) admits a unique solution $z\in L^\infty([0,+\infty),X)$,and $z=(u(t),u_t(t),\eta^t(s))$ depends continuously on initial data z_0 in X .

Proof:The existence of global solution is proved by Galerkin method.see[8, 9, 14].

Step1.Construct approximate solution

Let $(-\Delta)^{2m}\omega_i=\lambda_i^{2m}\omega_i$,where λ_i is the eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary on Ω , ω_i is the characteristic function corresponding to eigenvalue λ_i,According to the eigenvalue theory, $\omega_1,\omega_2,\cdots,\omega_l$ constitutes the standard orthogonal basis of H . Fixed $T>0$,For a given integer $\forall l\in\mathrm{N}$,Let P_l and Q_l denote the projection operator from the following space to its subspace, respectively:

$span\{\omega_1,\omega_2,\cdots,\omega_l\}\subset V_m, span\{\varsigma_1,\varsigma_2,\cdots,\varsigma_l\}\subset L^2_\mu\left(R^+;V_m\right),$

Let the approximate solution of problem(4)

be $u_l\left(t\right)=\sum_{i=1}^{l}u_{il}\left(t\right)\omega_i,\eta_l^t\left(s\right)=\sum_{i=1}^{l}\eta_{il}^t\left(s\right)\varsigma_i$,where $u_{il}\left(t\right),\eta_{il}^t\left(s\right)$ is determined by the following nonlinear ordinary differential equations

$$\begin{cases}\left(u_{ltt}+\left(-\Delta\right)^m u_{lt}+\phi\left(\left\|\nabla^m u_l\right\|^2\right)\left(-\Delta\right)^m u_l+\int_0^\infty\mu\left(s\right)\left(-\Delta\right)^m\eta_l^t\left(s\right)ds+f\left(u_l\right),\omega_i\right)=\left(h(x),\omega_i\right),\\ \left(\eta_t+\eta_s,\varsigma_i\right)_{\mathrm{E}}=\left(u,\varsigma_i\right)_{\mathrm{E}}, \qquad i=1,2,\cdots,l,\end{cases} \quad (20)$$

meets the initial conditions $z_{l0}=\left(u_{l0},u_{l1},\eta_{l0}\right)$,when $l\to+\infty$, $z_{l0}\to z_0$ in X . From the basic theory of ordinary differential equation, we know that the approximate solution $u_l\left(t\right),\eta_l^t\left(s\right)$ exists on $\left(0,t_l\right)$.

Step2.Prior estimation

Because it is necessary to prove the existence of solutions in X, (20) Multiply both ends by $u_{ilt}\left(t\right)+\varepsilon u_{il}\left(t\right)$,and sum over i ,let $v_l\left(t\right)=u_{lt}\left(t\right)+\varepsilon u_l\left(t\right)$,According to lemma 2.1,the prior estimation of solutions in X spaces is obtained:

$\left\|v_l\right\|^2+\left\|\nabla^m u_l\right\|^2+\left\|\eta_l^t\right\|_{\mathrm{E}}^2\le R_1^2$, (21)

We have $z=\left(u_l,v_l,\eta_l^t\right)$ is uniformly bounded in $L^\infty\left(\left[0,+\infty\right);X\right)$.

Step3.Limit process

because $\{u_l\}$ is bounded on V_m , $\{u_l\}$ has subsequences strongly convergent to u on H ,So there are subsequences still represented by $\{u_l\}$,such that

$\{u_l\}$ almost everywhere converges to u in H .

According to a priori estimate, we have

$\left(\left(-\Delta\right)^m u_{lt},\omega_i\right)=\left(v_l,\lambda_1^m\omega_i\right)-\left(\varepsilon u_l,\lambda_1^m\omega_i\right),$

so $\left(\left(-\Delta\right)^m u_{lt},\omega_i\right)\to\left(v,\lambda_1^m\omega_i\right)-\left(\varepsilon u,\lambda_1^m\omega_i\right)$ weak*in $L^\infty\left[0,+\infty\right)$.

By $\left(u_{lt},\omega_i\right)\to\left(u_t,\omega_i\right)$ weak*in $L^\infty\left[0,+\infty\right)$,

then $\left(u_{ltt},\omega_i\right)=\frac{d}{dt}\left(u_{lt},\omega_i\right)\to\left(u_{tt},\omega_i\right)$ in $D'\left[0,+\infty\right)$,where $D'\left[0,+\infty\right)$ is a conjugate space of $D\left[0,+\infty\right)$ infinitely differentiable spaces.

$\left(\int_0^\infty\mu\left(s\right)\left(-\Delta\right)^m\eta_l^t\left(s\right)ds,\omega_i\right)\to\left(\int_0^\infty\mu\left(s\right)\left(-\Delta\right)^m\eta^t\left(s\right)ds,\omega_i\right)$ weak*in $L^\infty\left[0,+\infty\right)$.

By the assumption $\left(\mathrm{I}\right)$, $\left(f\left(u_l\right),\omega_i\right)\to\left(f\left(u\right),\omega_i\right)$ weak*in $L^\infty\left[0,+\infty\right)$.

and,

$$\left(\phi\left(\left\|\nabla^m u_l\right\|^2\right)\left(-\Delta\right)^m u_l,\omega_i\right)=\phi\left(\left\|\nabla^m u_l\right\|^2\right)\left(\left(-\Delta\right)^{\frac{m}{2}}u_l,\left(-\Delta\right)^{\frac{m}{2}}\omega_i\right)$$

$$=\phi\left(\left\|\nabla^m u_l\right\|^2\right)\left(\left(-\Delta\right)^{\frac{m}{2}}u_l,\lambda_1^{\frac{m}{2}}\omega_i\right)\to\phi\left(\left\|\nabla^m u\right\|^2\right)\left(\left(-\Delta\right)^{\frac{m}{2}}u,\lambda_1^{\frac{m}{2}}\omega_i\right),$$

weak*in $L^\infty\left[0,+\infty\right)$.

In particular, $z_{l0}\to z_0$ weak in X . For all i and when $l\to+\infty$,According to the density of substrate $\omega_1,\omega_2,\cdots,\omega_l,\cdots$,we get

$$\left(u_{ltt}+(-\Delta)^m u_{lt}+\phi\left(\left\|\nabla^m u_l\right\|^2\right)(-\Delta)^m u_l+\int_0^\infty \mu(s)(-\Delta)^m \eta_l^t(s)ds+f(u_l),\omega\right)=(h(x),\omega),\ \forall\omega\in V_m,$$

Therefore, the existence of weak solutions to problem(4)-(6)is proved.

For $\forall t\in R$, let $z_i=\left(u_i(t),u_{it}(t),\eta_i^t(s)\right),(i=1,2)$ be two solutions of problem(4)-(6) as shown above corresponding to initial data

$z_0^i=\left(u_0^i,u_1^i,\eta_0^i(s)\right)$,Then $w(t)=u_1(t)-u_2(t)$, $\xi^t(s)=\eta_1^t(s)-\eta_2^t(s)$ satisfifies

$$\begin{cases} w_{tt}+(-\Delta)^m w_t+\phi\left(\left\|\nabla^m u_1\right\|^2\right)(-\Delta)^m u_1-\phi\left(\left\|\nabla^m u_2\right\|^2\right)(-\Delta)^m u_2 \\ +\int_0^\infty \mu(s)(-\Delta)^m \xi^t(x,s)ds+f(u_1)-f(u_2)=0, \\ \xi_t=-\xi_s+w_t, \\ \left(w(0),w_t(0),\xi^0(s)\right)=\left(u_0^1,u_1^1,\eta_0^1\right)-\left(u_0^2,u_1^2,\eta_0^2\right), \end{cases} \quad (22)$$

Taking H -inner product by w_t in (22) and making use of assumptions $(\mathrm{I})-(\mathrm{III})$,we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2}\|w_t\|^2+\frac{1}{2}\phi_0\left\|\nabla^m w\right\|^2+\frac{1}{2}\left\|\xi^t\right\|_{\mathrm{E}}^2\right]+\left\|\nabla^m w_t\right\|^2+\frac{\mu_1}{2}\left\|\xi^t\right\|_{\mathrm{E}}^2 \\ &=\left(\phi_0-\phi\left(\left\|\nabla^m u_1\right\|^2\right)\right)\left(\nabla^m w,\nabla^m w_t\right)-\left[\phi\left(\left\|\nabla^m u_1\right\|^2\right)-\phi\left(\left\|\nabla^m u_2\right\|^2\right)\right]\left((-\Delta)^m u_2,w_t\right) \quad (23) \\ &-\left(f(u_1)-f(u_2),w_t\right)\le\frac{1}{2}\left\|\nabla^m w_t\right\|^2+C(R_1)\left(\left\|\nabla^m w\right\|^2+\|w_t\|^2\right), \end{aligned}$$

then

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\|w_t\|^2+\phi_0\left\|\nabla^m w\right\|^2+\left\|\xi^t\right\|_{\mathrm{E}}^2\right]+\left\|\nabla^m w_t\right\|^2\le C(R_1)\left(\|w_t\|^2+\phi_0\left\|\nabla^m w\right\|^2+\left\|\xi^t\right\|_{\mathrm{E}}^2\right), \quad (24)$$

Applying the Gronwall inequality to(24)

$$\|w_t\|^2+\phi_0\left\|\nabla^m w\right\|^2+\left\|\xi^t\right\|_{\mathrm{E}}^2\le\left(\|w_1\|^2+\phi_0\left\|\nabla^m w_0\right\|^2+\left\|\xi^0\right\|_{\mathrm{E}}^2\right)e^{C(R_1)t}. \quad (25)$$

(25) implies that空间中,$\left(u,u_t,\eta^t\right)$depends continuously on initial data z_0 in X ,and hence, the solution of

Problem is unique.

Theorem 2.2 is proven.

3. Global Attractor

Lemma3.1[15] Let $H:R^+\to R^+$ be an absolutely continuous function,and

$$\frac{\mathrm{d}}{\mathrm{dt}}H(t)+2\delta H(t)\le h(t)H(t)+z(t),\ t>0,$$

where $\delta>0,\ z\in L_{loc}^1\left(R^+\right),\ h$ satisfies

$$\int_s^t h(\tau)\mathrm{d}\tau\le\delta(t-s)+m,\ t\ge s\ge 0,\ m>0.$$

Then

$$H(t)\le e^m\left(H(0)e^{-\delta t}+\int_0^t|z(\tau)|e^{-\delta(t-\tau)}\mathrm{d}\tau\right),\ t>0.$$

Lemma3.2[16] Let $\{S(t)\}_{t\ge0}$ be a semigroup on Banach Space $(X,\|\bullet\|)$, B is a bounded positive invariant set in X ,for $\forall\nu>0,\ \exists T=T(\nu,B)$,such that

$$\|S(T)x-S(T)y\|\le\nu+\Phi_T(x,y),\ \forall x,y\in B,$$

where $\Phi_T : X \times X \to R$,for $\forall x_n \subset B$,satisfies

$$\lim_{k\to\infty}\lim_{l\to\infty}\Phi_T\left(x_k, x_l\right) = 0 .$$

Then $\left\{S(t)\right\}_{t\ge 0}$ is asymptotically compact in X .

Lemma3.3[14] A dissipative dynamical system $\left(S(t), X\right)$ has a compact global attractor A if and only if it is asymptotically compact.

By Theorem2.2, the solution of problem(4)-(6)is well-posed in weak topological space, Thus, the mapping can be defined $S(t): X \to X$, i.e

$$S(t)\left(u_0,u_1,\eta^0\right)=\left(u(t),u_t(t),\eta^t\right),\ t\ge 0 ,$$

where $\left(u(t),u_t(t),\eta^t\right)$ is the unique weak solution of problem(4)-(6), $\left\{S(t)\right\}_{t\ge 0}$ satisfies the properties of semigroups and is a locally Lipschitz continuous nonlinear C_0 – semigroup on X .

Lemma3.4: Assumed that the assumptions of Theorem2.2,Then the corresponding semigroup $\left\{S(t)\right\}_{t\ge 0}$ of problem(4)-(6) has a bounded absorbing set B_0 in X .

Proof: The conclusion of lemma 3.4 can be obtained from the conclusion of lemma 2.1.

Let u is the solution of problem (4)-(6),and

$$B_0 = \bigcup_{t\ge 0} S(t) B_1 ,$$

where

$$B_1 = \left\{\left(u_0,u_1,\eta_0\right)\in X : \left\|\nabla^m u_0\right\|^2 + \left\|u_1\right\|^2 + \left\|\eta_0\right\|_{\mathrm{E}}^2 \le R_1^2\right\} .$$

Lemma3.5 Assumed that the assumptions of Theorem2.2,Then the corresponding semigroup $\left\{S(t)\right\}_{t\ge 0}$ of problem(4)-(6)is asymptotically compact in X .

Proof:Let u,v be two solutions of problem(4)-(6) as shown above corresponding to initial data $\left(u_0,u_1,\eta^0\right),\left(v_0,v_1,\zeta^0\right)$, respectively,Then $w=u-v,\ \xi^t=\eta^t-\zeta^t$ Satisfifies

$$\begin{cases} w_{tt}+(-\Delta)^m w_t+\phi_{12}(t)(-\Delta)^m w-\tilde{\phi}_{12}(t)\left(\nabla^m(u+v),\nabla^m w\right)(-\Delta)^m(u+v) \\ +\int_0^\infty \mu(s)(-\Delta)^m \xi^t(x,s)ds+f(u)-f(v)=0, \\ \xi_t=-\xi_s+w_t, \\ \left(w(0),w_t(0),\xi^0(s)\right)=\left(u_0,u_1,\eta_0\right)-\left(v_0,v_1,\zeta_0\right), \end{cases} \tag{26}$$

where $\phi_{12}(t)=\frac{1}{2}\left(\phi\left(\left\|\nabla^m u\right\|^2\right)+\phi\left(\left\|\nabla^m v\right\|^2\right)\right)>0$, $\tilde{\phi}_{12}(t)=\frac{1}{2}\int_0^1\phi'\left(\tau\left\|\nabla^m u\right\|^2+(1\text{-}\tau)\left\|\nabla^m v\right\|^2\right)d\tau>0$.

Taking H -inner product by w_t in (26) and making use of assumptions (II) , (III), we have

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\left\|w_t\right\|^2+\phi_{12}(t)\left\|\nabla^m w\right\|^2+\tilde{\phi}_{12}(t)\left(\nabla^m(u+v),\nabla^m w\right)^2\right]+\left\|\nabla^m w_t\right\|^2 \\ &+\left(\int_0^\infty\mu(s)(-\Delta)^m\xi^t(x,s)ds,w_t\right)=\frac{1}{2}\left(\phi'\left(\left\|\nabla^m u\right\|^2\right)\left(\nabla^m u,\nabla^m u_t\right)+\phi'\left(\left\|\nabla^m v\right\|^2\right)\left(\nabla^m v,\nabla^m v_t\right)\right)\left\|\nabla^m w\right\|^2 \\ &+\tilde{\phi}_{12}(t)\left(\nabla^m(u_t+v_t),\nabla^m w\right)\left(\nabla^m(u+v),\nabla^m w\right) \\ &+\frac{1}{2}\int_0^1\phi''\left(\tau\left\|\nabla^m u\right\|^2+(1-\tau)\left\|\nabla^m v\right\|^2\right)\left(\tau\left(\nabla^m u,\nabla^m u_t\right)+(1-\tau)\left(\nabla^m v,\nabla^m v_t\right)\right)d\tau\left(\nabla^m(u+v),\nabla^m w\right)^2 \\ &-\left(f(u)-f(v),w_t\right)\le C\bullet\left(\left\|\nabla^m u_t\right\|+\left\|\nabla^m v_t\right\|\right)\left\|\nabla^m w\right\|^2-\left(f(u)-f(v),w_t\right), \end{aligned} \tag{27}$$

Similarly, Taking H -inner product by w in (26) and making use of assumption (III) ,we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(w_t,w)+\frac{1}{2}\|\nabla^m w\|^2\right]+\phi_{12}(t)\|\nabla^m w\|^2+\tilde{\phi}_{12}(t)\left(\nabla^m(u+v),\nabla^m w\right)^2$$
$$+\left(\int_0^\infty \mu(s)(-\Delta)^m \xi^t(x,s)ds,w\right)\le \lambda_1^{-m}\|\nabla^m w_t\|^2-\left(f(u)-f(v),w\right), \quad (28)$$

By $(\mathrm{I}),(\mathrm{III})$,

$$\left(f(u)-f(v),w_t\right)\le C\int_\Omega(1+|u|+|v|)|w||w_t|dx\le C\|w\|\|w_t\|+C\left(\|u\|_3+\|v\|_3\right)\|w\|_3\|w_t\|_3$$
$$\le \varepsilon\|\nabla^m w_t\|^2+C\|\nabla^m w\|^2,$$
$$\left(f(u)-f(v),w\right)\le C\int_\Omega(1+|u|+|v|)|w|^2dx\le C\|w\|^2+C\left(\|u\|_3+\|v\|_3\right)\|w\|_3^2\le C\|w\|^2+C\|\nabla^m w\|^2,$$
$$\left(\int_0^\infty \mu(s)(-\Delta)^m \xi^t(x,s)ds,w_t\right)\ge \frac{1}{2}\frac{d}{dt}\|\xi^t\|_{\mathrm{E}}^2+\frac{\mu_1}{2}\|\xi^t\|_{\mathrm{E}}^2,$$
$$\left(\int_0^\infty \mu(s)(-\Delta)^m \xi^t(x,s)ds,w\right)\ge -\frac{1}{4}\|\xi^t\|_{\mathrm{E}}^2+\frac{\mu_0}{\mu_1}\|\nabla^m w\|^2,$$

$(27)+\varepsilon(28)$:

$$\frac{d}{dt}H_2(t)+K_2(t)\le C\bullet\left(\|\nabla^m u_t(t)\|+\|\nabla^m v_t(t)\|\right)\|\nabla^m w(t)\|^2+C\|w(t)\|^2, \quad (29)$$

When $0<\varepsilon\le\min\left\{1,\frac{\phi_0}{1+\lambda_1^{-m}},\frac{1}{2+\lambda_1^{-m}},\frac{\mu_1}{2+\mu_0}\right\}$,we obtain

$$C(\varepsilon)\left(\|w_t\|^2+\|\nabla^m w\|^2+\|\xi^t\|_{\mathrm{E}}^2\right)$$
$$\le H_2(t)=\frac{1}{2}\left(\|w_t\|^2+\phi_{12}(t)\|\nabla^m w\|^2+\tilde{\phi}_{12}(t)\left(\nabla^m(u+v),\nabla^m w\right)^2+\|\xi^t\|_{\mathrm{E}}^2+\varepsilon\|\nabla^m w\|^2\right)+\varepsilon(w_t,w)$$
$$\le\|w_t\|^2+\phi_{12}(t)\|\nabla^m w\|^2+\tilde{\phi}_{12}(t)\left(\nabla^m(u+v),\nabla^m w\right)^2+\|\xi^t\|_{\mathrm{E}}^2,$$
$$K_2(t)=\left(1-\varepsilon-\varepsilon\lambda_1^{-m}\right)\|\nabla^m w_t\|^2+\left(\frac{\mu_1}{2}-\varepsilon\frac{\mu_0}{2}\right)\|\xi^t\|_{\mathrm{E}}^2+\varepsilon\phi_{12}(t)\|\nabla^m w\|^2+\varepsilon\tilde{\phi}_{12}(t)\left(\nabla^m(u+v),\nabla^m w\right)^2$$
$$\ge\alpha_2\left(\|w_t\|^2+\|\xi^t\|_{\mathrm{E}}^2+\phi_{12}(t)\|\nabla^m w\|^2+\tilde{\phi}_{12}(t)\left(\nabla^m(u+v),\nabla^m w\right)^2\right),$$

where $\alpha_2=\min\left\{\varepsilon,\lambda_1^m\varepsilon\right\}$,then

$$K_2(t)-\alpha_2 H_2(t)\ge 0, \quad (30)$$

Substituting (30)into(29),we have

$$\frac{d}{dt}H_2(t)+\alpha_2 H_2(t)\le C\bullet\left(\|\nabla^m u_t(t)\|+\|\nabla^m v_t(t)\|\right)H_2(t)+C\|w(t)\|^2, \quad (31)$$

By(8),we have

$$C\int_s^t\|\nabla^m u_t(\tau)\|d\tau\le C\left(\int_s^t\|\nabla^m u_t(\tau)\|^2d\tau\right)^{\frac{1}{2}}(t-s)^{\frac{1}{2}}\le\frac{\alpha_2}{2}(t-s)+C,$$

From lemma 2.1 and lemma 3.1, it is concluded that

$$\|w_t\|^2+\|\nabla^m w\|^2+\|\xi^t\|_{\mathrm{E}}^2\le C\left(\|w_1\|^2+\|\nabla^m w_0\|^2+\|\xi^0\|_{\mathrm{E}}^2\right)e^{-\alpha_2 t}+C\int_0^t e^{-\alpha_2(t-\tau)}\|w(\tau)\|^2d\tau. \quad (32)$$

Choose a big enough T to make

$$C\left(\|w_1\|^2+\|\nabla^m w_0\|^2++\|\xi^0\|_{\mathrm{E}}^2\right)e^{-\alpha_2 t}\le\nu,$$

Let $\Phi_T\left((u_0,u_1,\eta_0),(v_0,v_1,\zeta_0)\right)=C\int_0^t e^{-\alpha_2(t-\tau)}\|w(\tau)\|^2d\tau$,then

Fot all $(u_0,u_1,\eta_0),(v_0,v_1,\zeta_0)\in B_0$,we have

$\|S(T)(u_0,u_1,\eta_0)-S(T)(v_0,v_1,\zeta_0)\|\le \nu+\Phi_T((u_0,u_1,\eta_0),(v_0,v_1,\zeta_0))$,

by $(u_0^n,u_1^n,\eta_0^n)\subset B_0$,and B_0 is a bounded positive invariant set,Then the solution (u^n,u_t^n,η^n) of problem (4)-(6) is uniformly bounded in X ,moreover $\{u^n\}$ is bounded in $C([0,\infty);V_m)\cap C^1([0,\infty);H)$.

Since $V_m\mapsto H$ is tight embedding,There exists a sequence $\{u_k^n\}$ which is strongly convergent in $C([0,T];H)$,and

$\lim_{k\to\infty}\lim_{l\to\infty}\Phi_T((u_{k0}^n,u_{k1}^n,\eta_{k0}^n),(u_{l0}^n,u_{l1}^n,\eta_{l0}^n))=0$,

Then we have $\{S(t)\}_{t\ge 0}$ is asymptotically compact in X .

Lemma 3.5 is proved.

Theorem3.6 Assumed that the assumptions of Theorem2.2,Then the semigroup $\{S(t)\}_{t\ge 0}$ possesses in X a global

attractor A which is connected.

Proof: By lemma 3.4 and lemma 3.5, $(S(t),X)$ is a dissipative dynamic system and is asymptotically compact, According to lemma 3.3, we know that there is a compact global attractor A.

Theorem 3.6 is proved.

4. Exponential Attractor

Definition4.1[15] Semi modules $n(\bullet)$ on Banach spaces X is called compact semimodules, If for any bounded set $B\subset X$,there is a sequence $\{x_n\}\subset B$ such that when $n,m\to\infty$, $n(x_m-x_n)\to 0$.

Definition 4.2[14] Set $A_{\exp}$ in complete metric space X is called exponential attractor of semigroup $\{S(t)\}_{t\ge 0}$,If the following conditions are met

①$A_{\exp}$ is a compact set in X ;

②$A_{\exp}$ has finite fractal dimension in X ;

③$A_{\exp}$ is a positive invariant set, i.e $S(t)A_{\exp}\subset A_{\exp}$ for $\forall t>0$;

④$A_{\exp}$ attracts bounded sets in X with exponential rate,i.e there is a constant $\sigma>0$ such that for any bounded set $B\subset X$ and any $t>0$ satisfy

$dist_X\{S(t)B,A_{\exp}\}\le C(B)e^{-\sigma t}$.

Lemma4.3[9, 15] Let B be a bounded closed set in Banach Space X . If mapping $F:B\to B$ satisfies

(1) F Lipschitz continuous on B , i.e for $\forall u_1,u_2\in B$, $\exists L>0$ such that

$\|Fu_1-Fu_2\|\le L\|u_1-u_2\|$;

(2) There are compact semimodules $n_1(x),n_2(x)$ in X ,and $\exists 0<\theta<1, K>0$ such that for $\forall u_1,u_2\in B$ satifies

$\|Fu_1-Fu_2\|\le\theta\|u_1-u_2\|+K[n_1(u_1-u_2)+n_2(Fu_1-Fu_2)]$;

Then for $\forall\kappa>0,\delta\in(0,1-\theta)$,Positive invariant compact set $A_{q,\kappa}\subset B$ with finite fractal dimension:

$$\sup\left\{F^kB, A_{q,\kappa}\right\} = \sup_{u\in B}\left\{dist\left(F^ku, A_{q,\kappa}\right)\right\} \le q^k,\ k = 1,2,\cdots,$$

where $q = \theta + \delta < 1$,and

$$\dim_f A_{q,\kappa} \le \left(\ln\frac{1}{\delta+\theta}\right)^{-1} \bullet \left(\ln m_0\left(\frac{2K\left(1+L^2\right)^{1/2}}{\delta}\right) + \kappa\right),$$

where $m_0(R)$ is the maximum number of points (x_i, y_i) in product space $X\times X$ satisfying the following conditions:

$$\|x_i\|^2 + \|y_i\|^2 \le R^2,\ n_1\left(x_i - x_j\right) + n_2\left(y_i - y_j\right) > 1,\ i \ne j,$$

i.e $A_{q,\kappa}$ is the exponential attractor of discrete dynamical system $\left(F^k, B\right)$.

Lemma4.4[15] Let X, Y be metric space, If $q: X \to Y$ is a $\alpha - Hölder$ continuous mapping, then

$$\dim_f\left\{q(Q), Y\right\} \le \frac{1}{\alpha}\dim_f\left\{Q, X\right\}.$$

Theorem4.5 Assumed that the assumptions of Theorem2.2,Then the dynamical system $\left(S(t), X\right)$ has an exponential attractor $A_{\exp}$.

Proof: Differentiating for t in equation(1),let $v(t) = u_t(t)$,satisfies

$$\begin{aligned} &v_{tt} + (-\Delta)^m v_t + \phi\left(\|\nabla^m u\|^2\right)(-\Delta)^m v + 2\phi'\left(\|\nabla^m u\|^2\right)\left(\nabla^m u, \nabla^m u_t\right)(-\Delta)^m u \\ &+\int_0^\infty g(s)(-\Delta)^m v_t(t-s)\,ds + f'(u)v = 0, \end{aligned} \tag{33}$$

Let $\eta_v = \eta_v^t(x,s) = \int_0^s v_t(x, t-\tau)\,d\tau,\ (x,s)\in\Omega\times R^+, t\ge 0$. (34)

By formal differentiation we have

$\eta_{vt}^t(x,s) = -\eta_{vs}^t(x,s) + v_t(x,t),\ (x,s)\in\Omega\times R^+, t\ge 0$. (35)

Then equation (33) can be transformed into the following equivalent autonomous system:

$$\begin{cases} v_{tt} + (-\Delta)^m v_t + \phi\left(\|\nabla^m u\|^2\right)(-\Delta)^m v + 2\phi'\left(\|\nabla^m u\|^2\right)\left(\nabla^m u, \nabla^m u_t\right)(-\Delta)^m u \\ +\int_0^\infty \mu(s)(-\Delta)^m \eta_v^t(s)\,ds + f'(u)v = 0, \\ \eta_{vt} = -\eta_{vs} + v_t,\ (x,t,s)\in\Omega\times R^+\times R^+,\quad (x,t)\in\Omega\times[0,\infty). \end{cases} \tag{36}$$

Taking H -inner product by $v_t + \varepsilon v$ in (36) and making use of assumptions $(\mathrm{I})-(\mathrm{III})$ and(7),we have

$$\begin{aligned} &\frac{d}{dt}H_3(t) + \|\nabla^m v_t\|^2 - \varepsilon\|v_t\|^2 + \varepsilon\phi\left(\|\nabla^m u\|^2\right)\|\nabla^m v\|^2 - \varepsilon^2\frac{\mu_0}{\mu_1}\|\nabla^m v\|^2 + \frac{\mu_1}{4}\|\eta_v^t\|_{\mathrm{E}}^2 \\ &+2\varepsilon\phi'\left(\|\nabla^m u\|^2\right)\left(\nabla^m u, \nabla^m v\right)^2 \le \phi'\left(\|\nabla^m u\|^2\right)\left(\nabla^m u, \nabla^m u_t\right)\|\nabla^m v\|^2 \\ &-2\phi'\left(\|\nabla^m u\|^2\right)\left(\nabla^m u, \nabla^m u_t\right)\left(\nabla^m u, \nabla^m v_t\right) - \left(f'(u)v, v_t + \varepsilon v\right) \\ &\le \frac{1}{2}\|\nabla^m v_t\|^2 + \varepsilon\|v_t\|^2 + C\|\nabla^m u_t\|\|\nabla^m v\|^2 + C\|\nabla^m v\|^2, \end{aligned} \tag{37}$$

where

$$H_3(t) = \frac{1}{2}\left(\|v_t\|^2 + \phi\left(\|\nabla^m u\|^2\right)\|\nabla^m v\|^2 + \varepsilon\|\nabla^m v\|^2 + \|\eta_v^t\|_{\mathrm{E}}^2\right) + \varepsilon\left(v_t, v\right),$$

Then

$$\frac{d}{dt}H_3(t)+K_3(t)\le C\left\|\nabla^m u_t\right\|\left\|\nabla^m v\right\|^2+C\left\|\nabla^m v\right\|^2, \quad (38)$$

When $0<\varepsilon\le\min\left\{1,\frac{\phi_0}{1+\lambda_1^{-m}},\frac{\lambda_1^m}{4},\frac{\phi_0\mu_1}{2\mu_0}\right\}$,

$$C(\varepsilon)\left(\left\|v_t\right\|^2+\left\|\nabla^m v\right\|^2+\left\|\eta_v^t\right\|_{\mathrm{E}}^2\right)\le H_3(t)\le\left\|v_t\right\|^2+\phi\left(\left\|\nabla^m u\right\|^2\right)\left\|\nabla^m v\right\|^2+\left\|\eta_v^t\right\|_{\mathrm{E}}^2,$$

$$K_3(t)=\frac{1}{2}\left\|\nabla^m v_t\right\|^2-2\varepsilon\left\|v_t\right\|^2+\varepsilon\phi\left(\left\|\nabla^m u\right\|^2\right)\left\|\nabla^m v\right\|^2-\varepsilon^2\frac{\mu_0}{\mu_1}\left\|\nabla^m v\right\|^2+\frac{\mu_1}{4}\left\|\eta_v^t\right\|_{\mathrm{E}}^2$$

$$+2\varepsilon\phi'\left(\left\|\nabla^m u\right\|^2\right)\left(\nabla^m u,\nabla^m v\right)^2\ge\alpha_3\left(\left\|v_t\right\|^2+\phi\left(\left\|\nabla^m u\right\|^2\right)\left\|\nabla^m v\right\|^2+\left\|\eta_v^t\right\|_{\mathrm{E}}^2\right),$$

where $\alpha_3=\min\left\{\frac{\lambda_1^m}{2}-2\varepsilon,\frac{\varepsilon}{2},\frac{\mu_1}{4}\right\}$,we get

$$K_3(t)-\alpha_3H_3(t)\ge 0,$$

so

$$\frac{d}{dt}H_3(t)+\alpha_3H_3(t)\le C\left\|\nabla^m u_t\right\|H_3(t)+C\left\|\nabla^m v\right\|^2, \quad (39)$$

From lemma 3.1, we get

$$\left\|v_t\right\|^2+\left\|\nabla^m v\right\|^2+\left\|\eta_v^t\right\|_{\mathrm{E}}^2\le Ce^{-\alpha_3 t}+C.$$

and

$$\left\|u_{tt}\right\|^2+\left\|\nabla^m u_t\right\|^2+\left\|\eta_v^t\right\|_{\mathrm{E}}^2\le C. \quad (40)$$

From lemma 3.4, we know that dynamical system $\left(S(t),X\right)$ has a closed positive invariant bounded absorbing set B_0 ,by(7)and(40),we have B_0 is bounded in $V_m\times V_m\times\mathrm{E}$,and for any $y_u=\left(u_0,u_1,\eta_0^t\right)\in B_0$, $y_u(t)=S(t)y_u=\left(u(t),u_t(t),\eta^t\right)\in B_0$ satisfies

$$\left\|u_{tt}\right\|^2+\left\|\nabla^m u_t\right\|^2+\left\|\eta^t\right\|_{\mathrm{E}}^2+\left\|\eta_v^t\right\|_{\mathrm{E}}^2\le C, \quad (41)$$

Defining operator:

$$F=S(T):B_0\to B_0$$

obviously $FB_0\subset B_0$,and easily know F is a Lipschitz operator,

Taking H -inner product by $w_t+\varepsilon w$ in (26),and making use of assumptions (III)

$$\frac{d}{dt}H_4(t)+\left\|\nabla^m w_t(t)\right\|^2\le K_4(t)-\left(f(u)-f(v),w_t+\varepsilon w\right), \quad (42)$$

when $\varepsilon>0$ is appropriately small,

$$H_4(t)=\frac{1}{2}\left(\left\|w_t\right\|^2+\varepsilon\left\|\nabla^m w\right\|^2+\left\|\xi^t\right\|_{\mathrm{E}}^2\right)+\varepsilon\left(w_t,w\right)\ge C(\varepsilon)\left(\left\|w_t\right\|^2+\left\|\nabla^m w\right\|^2+\left\|\xi^t\right\|_{\mathrm{E}}^2\right),$$

and

$$K_4=-\phi_{12}(t)\left(\nabla^m w,\nabla^m w_t\right)-\tilde{\phi}_{12}(t)\left(\nabla^m(u+v),\nabla^m w_t\right)\left(\nabla^m(u+v),\nabla^m w\right)$$

$$-\varepsilon\left(\phi_{12}(t)\left\|\nabla^m w\right\|^2+\tilde{\phi}_{12}(t)\left(\nabla^m(u+v),\nabla^m w\right)^2\right)+\varepsilon\left\|w_t\right\|^2-\frac{1}{4}\left\|\xi^t\right\|_{\mathrm{E}}^2 \quad (43)$$

$$\le\frac{1}{8}\left\|\nabla^m w_t\right\|^2+C\left(\left\|\nabla^m w\right\|^2+\left\|\xi^t\right\|_{\mathrm{E}}^2\right),$$

obviously

$$\left|\left(f(u)-f(v),w_t+\varepsilon w\right)\right|\le C\int_\Omega\left(1+|u|+|v|\right)|w|\left(|w_t|+\varepsilon|w|\right)dx$$
$$\le C\|w\|\left(\|w_t\|+\varepsilon\|w\|\right)+C\left(\|u\|_3+\|v\|_3\right)\|w\|_3\left(\|w_t\|_3+\varepsilon\|w\|_3\right) \qquad (44)$$
$$\le\frac{1}{8}\left\|\nabla^m w_t\right\|^2+C\left\|\nabla^m w\right\|^2,$$

Substituting (43)(44)into(42),we get

$$\frac{d}{dt}H_4(t)+\frac{1}{2}\left\|\nabla^m w_t(t)\right\|^2\le CH_4(t),$$

Using Gronwall inequality, we get

$$\|w_t\|^2+\left\|\nabla^m w\right\|^2+\left\|\xi^t\right\|_{\mathrm{E}}^2+\int_0^t\left\|\nabla^m w(\tau)\right\|^2d\tau\le Ce^{\alpha_4 t}\left(\|w_1\|^2+\left\|\nabla^m w_0\right\|^2+\left\|\xi^0\right\|_{\mathrm{E}}^2\right).$$

i.e F is a Lipschitz operator.

For any $y_u,y_v\in B_0$,by(32),we have

$$\left\|Fy_u-Fy_v\right\|_X^2\le C\left\|y_u(t_0)-y_v(t_0)\right\|_X^2e^{-\alpha_2(T-t_0)}+C\int_{t_0}^Te^{-\alpha_2(T-\tau)}\left\|u(\tau)-v(\tau)\right\|^2d\tau$$
$$\le\theta_T^2\left\|y_u(t_0)-y_v(t_0)\right\|_X^2+C\max_{t_0\le s\le T}\left\|u(s)-v(s)\right\|^2,$$

i.e

$$\left\|Fy_u-Fy_v\right\|_X\le\theta_T\left\|y_u(t_0)-y_v(t_0)\right\|_X+Cn_1\left(y_u-y_v\right),$$

where

$$\theta_T^2=Ce^{-\alpha_2(T-t_0)},\qquad n_1\left(y_u-y_v\right)=\max_{t_0\le s\le T}\left\|u(s)-v(s)\right\|,$$

Easily know $n_1\left(y_u-y_v\right)$ is a compact semimodule in X .

According to lemma 4.3, there is an exponential attractor A_k in discrete dynamical system $\left(F^k,B_0\right)$,here $F^k=S(kT)$. Let

$$A_{\exp}=\bigcup_{0\le t\le T}S(t)A_k,$$

Combined with[17],we get $A_{\exp}$ is the exponential attractor of continuous dynamical system $\left(S(t),B_0\right)$,Therefore, according to the definition of exponential attractor, there is $\kappa>0$,such that

$$dist_X\left\{S(t)B,A_{\exp}\right\}\le Ce^{-\kappa t},t\ge 0.$$

In fact,

1). $A_{\exp}$ is positive invariant.

2). For any bounded B in X ,there exists $t_B>0$,such that $\forall t\ge t_B$,there holds $S(t)B\subset B_0$,then

$$dist_X\left\{S(t)B,A_{\exp}\right\}\le dist_X\left\{S(t-t_B)B_0,A_{\exp}\right\}\le C\left(\|B\|_X\right)e^{-\kappa t}.$$

And when $t\le t_B$,

$$dist_X\left\{S(t)B,A_{\exp}\right\}\le Ce^{\kappa t}e^{-\kappa t}\le C\left(\|B\|_X\right)e^{-\kappa t}.$$

3). Define operator

$$V:[0,T]\times A_k\to B_0,V\left(t,y_u\right)=y_u(t)=S(t)y_u,y_u\in A_k,$$

For any $y_u,y_{u1},y_{u2}\in A_k,t,t_1,t_2\in[0,T]$ satisfy

$$\left\|V\left(t_1,y_u\right)-V\left(t_2,y_u\right)\right\|_X\le\left|\int_{t_1}^{t_2}\left\|y_u'(\tau)\right\|_Xd\tau\right|\le\left(\int_{t_1}^{t_2}\left\|y_u'(\tau)\right\|_X^2d\tau\right)^{\frac{1}{2}}\left|t_1-t_2\right|^{\frac{1}{2}}\le C\left|t_1-t_2\right|^{\frac{1}{2}},$$
$$\left\|V\left(t,y_{u1}\right)-V\left(t,y_{u2}\right)\right\|_X=\left\|S(t)y_{u1}-S(t)y_{u2}\right\|_X\le C\left\|y_{u1}-y_{u2}\right\|_X,$$

i.e V is about t being $\frac{1}{2}-H\ddot{o}lder$ continuous,and about y_u being Lipschitz continuous,so

$A_{\exp} = V\{[0,T]\times A_k\}$ (The image of $[0,T]\times A_k$ in V)i s a compact set in X .

4). From lemma 4.4, we know that

$\dim_f \{A_{\exp}, X\} \le 2 + 2\dim_f \{A_k, X\} < +\infty$.

According to the definition4.2 of exponential attractor, we get $A_{\exp}$ is the exponential attractor of dynamical system $(S(t), X)$.

Theorem 4.5 is proved.

Note: theorem 4.5 shows that the global attractor A in theorem 3.6 has finite fractal dimension.

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